例題：　２次元確率変数(X,Y)の同時確率密度関数

$f(x,y)=\frac{1}{4}(x+2y)$ $(0\leqq x\leqq 2, 0\leqq y\leqq 1)$

(1)X,Y の周辺確率分布関数

$f_X(x)=\frac{1}{4}(x+1)$, $f_Y(y)=\frac{1}{2}(1+2y)$

(2)期待値 E[X],E[Y],E[X+Y],E[XY]の導出

●$E[X]=\int_0^2 x\,f_X(x)dx=\frac{1}{4}\int_0^2 x\,(x+1)dx=\frac{1}{4}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^2=\frac{7}{6}$

●$E[Y]=\int_0^1 y\,f_Y(y)dy=\frac{1}{2}\int_0^1 y\,(1+2y)dx=\frac{1}{2}\left[\frac{y^2}{2}+\frac{2y^3}{3}\right]_0^1=\frac{7}{12}$

●$E[X+Y]=\int_0^2\int_0^1(x+y)f(x,y)dydx=\frac{1}{4}\int_0^2\int_0^1(x+y)(x+2y)dydx=\frac{1}{4}\int_0^2\int_0^1(x^2+3xy+2y^2)dydx$

$=\frac{1}{4}\int_0^2\left[x^2y+\frac{3xy^2}{2}+\frac{2y^3}{3}\right]_0^1 dx=\frac{1}{4}\int_0^2\left(x^2+\frac{3x}{2}+\frac{2}{3}\right)dx=\frac{1}{4}\left[\frac{x^3}{3}+\frac{3x^2}{4}+\frac{2x}{3}\right]_0^2=\frac{1}{4}\left(\frac{8}{3}+3+\frac{4}{3}\right)=\frac{7}{4}$

●$E[XY]=\int_0^2\int_0^1 xyf(x,y)dydx=\frac{1}{4}\int_0^2\int_0^1 xy(x+2y)dydx=\frac{1}{4}\int_0^2\left[\frac{x^2y^2}{2}+\frac{2xy^3}{3}\right]_0^1 dx=\frac{1}{4}\int_0^2\left(\frac{x^2}{2}+\frac{2x}{3}\right)dx$

$=\frac{1}{4}\left[\frac{x^3}{6}+\frac{x^2}{3}\right]_0^2=\frac{2}{3}$

(3)期待値 E[X²],E[Y²],の導出

●$E[X^2]=\int_0^2 x^2\,f_X(x)dx=\frac{1}{4}\int_0^2 x^2\,(x+1)dx=\frac{1}{4}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_0^2=\frac{5}{3}$

●$E[Y^2]=\int_0^1 y^2\,f_Y(y)dy=\frac{1}{2}\int_0^1 y^2\,(1+2y)dx=\frac{1}{2}\left[\frac{y^3}{3}+\frac{y^4}{2}\right]_0^1=\frac{5}{12}$

分散 V[X],V[Y],Cov[X,Y]の導出

●$V[X]=E[X^2]-E[X]^2=\frac{5}{3}-\left(\frac{7}{6}\right)^2=\frac{11}{36}$

●$V[Y]=E[Y^2]-E[Y]^2=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{11}{144}$

●$Cov[X,Y]=E[XY]-E[X]E[Y]=\frac{2}{3}-\frac{7}{6}\,\frac{7}{12}=\frac{-1}{72}$